Optimal control of execution costs

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Part 1

basic models and implementations

Outline

- Basic model (introduction)
- Market impact / Price dynamics
- Optimal trading Strategy
- model with information, model with temporary impact
- Implementation

Basic model

- to acquire a large block of $\bar{S}$ shares over a fixed time interval [0,T]
- denote S_t be the shares acquired in period t at Price P_t
- impose a no-sales constraints ($S_t \geq 0$)

$$Min\ E[\sum_{t=1}^{T} P_t S_t]\ , \quad \text{where } \sum_{t=1}^{T} S_t = \bar{S}$$

- P_t includes 2 distinct components : price dynamics (absence of our trade) and impact of our trade, where the impact is a linear function of trade size(θ)
- there exists white noise, which is assumed to be a zero mean and IID random shock

$$P_t = P_{t-1} + \theta S_t + \varepsilon_t, \quad \theta > 0,\ \mathrm{E}[\varepsilon_t | S_t, P_{t-1}] = 0$$

Cont.

- dynamic programming algorithm : finding the optimal control $\{S_t^*\}$
- the optimal solution $\{S_1^*, S_2^*, \dots, S_T^*\}$ must also be optimal for remaining program at every intermediate date t.
- Bellman equation:

$$V_t\left(P_{t-1,} W_t\right) = Min\ E_t[P_t S_t + V_{t+1}(P_t, W_{t+1})]$$

- W_t: the number of shares remain to be purchased

$$W_t = W_{t-1} - S_{t-1}, \ \ W_1 = \bar{S}, \ W_{T+1} = 0$$

$P_t = P_{t-1} + \theta S_t + \varepsilon_t, \quad \theta > 0, \mathrm{E}[\varepsilon_t | S_t, P_{t-1}] = 0$

Best execution strategy(linear)

1. $V_t(P_{t-1,} W_t) = Min\ E_t[P_t S_t + V_{t+1}(P_t, W_{t+1})]$
2. t = T

$$V_T(P_{T-1,} W_T) = Min\ E_T[P_T \underbrace{W_T}_{}] = (P_{T-1} + \theta W_T)\, W_T$$

$= S_T$

3. t = T-1

$$V_{T-1}(P_{T-2,} W_{T-1}) = Min\ E_{T-1}[P_{T-1} S_{T-1} + V_T(P_{T-1}, W_{\mathrm{T}})]$$

4. substitute $V_T(P_{T-1}, W_{\mathrm{T}})$ and differentiate V_{T-1} with respect to S_{T-1}

$$\mathrm{S}^*_{\mathrm{T}-1} = W_{T-1}/2$$

$$V_{T-1}(P_{T-2,} W_{T-1}) = W_{T-1}\left(P_{T-2} + \frac{3}{4}\theta W_{T-1}\right)$$

Best execution strategy(linear)

- continue in this fashion:

$$\mathrm{S}^*_{\mathrm{T}-k} = W_{T-k}/(\mathrm{k+1})$$

$$V_{T-k}\left(P_{T-k-1,} W_{T-k}\right) = W_{T-k}\left(P_{T-k-1} + \frac{k+2}{2(k+1)}\theta W_{T-k}\right)$$

- until we reach the beginning of the program:

$$\mathrm{S}^*_1 = W_1/\mathrm{T}\,, \quad V_1\left(P_{0,} W_1\right) = W_1\left(P_0 + \frac{T+1}{2T}\theta W_1\right)$$

- $W_1 = \bar{S}$

$$\mathrm{S}^*_1 = \bar{S}/\mathrm{T}\,, \quad V_1\left(P_{0,} W_1\right) = P_0\bar{S} + \frac{\theta\bar{S}^2}{2}(1 + \frac{1}{T})$$

- forward substitution

$$\mathrm{S}^*_1 = \mathrm{S}^*_2 = \cdots = \mathrm{S}^*_T = \bar{S}/\mathrm{T}$$

Best execution strategy(linear)

- Observe $V_1\left(P_{0,} W_1\right) = P_0\bar{S} + \frac{\theta\bar{S}^2}{2}(1 + \frac{1}{T})$:

  no-impact costs $P_0\bar{S}$ & cumulative price impact $\frac{\theta\bar{S}^2}{2}(1 + \frac{1}{T})$
- impact term is a decreasing function of T : seems that impact become negligible if there is no time limit
- However, law of motion for P_t implies that the price impact θS_t of an individual trade has a **permanent effect** on P_t

Linear price impact with information

- X_t : a serially−correlated state variable which also affects the execution
- rewrite:

  $$P_t = P_{t-1} + \theta S_t + \gamma X_t + \varepsilon_t, \ \theta>0$$

  $$X_t = \rho X_{t-1} + \eta_t, \ \rho\epsilon(-1,1)$$

  where ε_t, η_t are independent white noise processes with mean 0
- X_t might be **public information**, e.g., S&P 500 index, which γ measures the sensitivity to markets movements(CAPM)
- X_t might be **private information**, which γ represents the importance of that information for P_t

Linear price impact with information

- best-execution strategy with information X_t

$$S^*_{T-k} = \delta_{w,\,k} W_{T-k} + \delta_{x,\,k} X_{T-k}\,,$$

affected by W_t and X_t

$$V_{T-k}(P_{T-k-1}, X_{T-k}, W_{T-k}) = P_{T-k-1} W_{T-k} + a_k W^2_{T-k} + b_k X_{T-k} W_{T-k} + c_k X^2_{T-k} + d_k$$

for $k = 0, 1, \ldots, T-1$, where

$$\delta_{w,\,k} \equiv \frac{1}{k+1}, \quad \delta_{x,\,k} \equiv \frac{\rho b_{k-1}}{2a_{k-1}}$$

and

$$a_k = \frac{\theta}{2}\left(1 + \frac{1}{k+1}\right), \qquad a_0 = \theta$$

positive

$$c_k = \rho^2 c_{k-1} - \frac{\rho^2 b^2_{k-1}}{4a_{k-1}}, \qquad c_0 = 0,$$

negative

$$b_k = \gamma + \frac{\theta \rho b_{k-1}}{2a_{k-1}}, \qquad b_0 = \gamma,$$

positive

$$d_k = d_{k-1} + c_{k-1}\sigma^2_\eta, \qquad d_0 = 0.$$

negative

implementation

- Japan Tobacco Inc., 2914.T
- 2022/06/24 – 2022/06/30(5 days)
- Only discuss ‘buying’
- Step:
  1. Decide parameters
  2. performance of strategy with and without information

$$P_t = P_{t-1} + \theta S_t + \gamma X_t + \varepsilon_t, \ \theta>0$$
$$X_t = \rho X_{t-1} + \eta_t, \ \rho \epsilon (-1,1)$$

$$P_t = P_{t-1} + \theta S_t + \gamma X_t + \varepsilon_t$$

X_t *and* ρ

- X_t : a serially−correlated state variable which also affects the execution

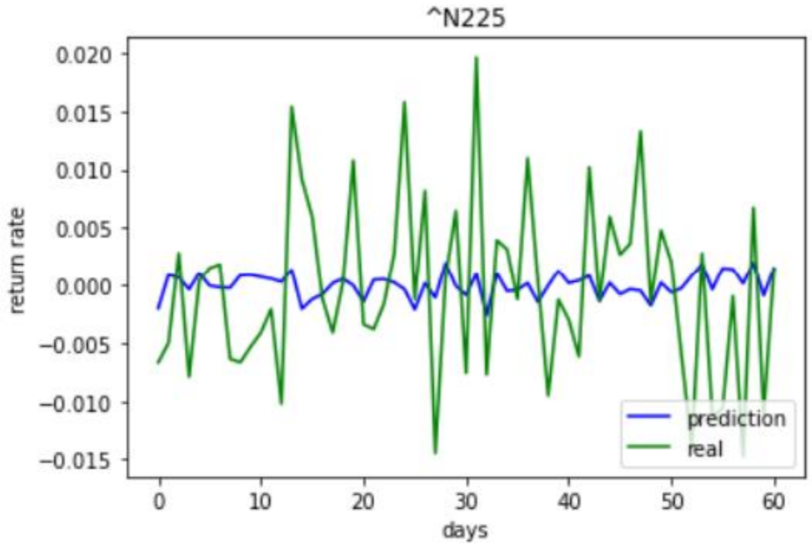
$$X_t = \rho X_{t-1} + \eta_t,\ \ \rho\epsilon(-1,1)$$

- set X_t daily return of ^N225 (2022/03/23-2022/06/23)

X_t=(^N225['Close']-^N225['Open'])/(^N225['Open'])

- $\rho = \frac{\frac{1}{T_1-1}\sum_{t=2}^{T_1}\widetilde{x_t}\,\widetilde{x_{t-1}}}{\frac{1}{T_1-1}\sum_{t=1}^{T_1-1}\widetilde{x_t}^{\,2}}$, $\widetilde{x_t} = (x_t - \mu)/\sigma$, $T_1 = 63\ (3\ months)$

- get ρ = -0.13356



Ref of AR(1) : http://www.liuyanecon.com/wp content/uploads/TS19Lec7.pdf?fbclid=IwAR0pL6jkBDYggIlyXj3f5zQIBW7JD9oaIc62ggIpYpA28mMLhCP-ZjDj5G4

S_t (size of meta order)

- identify order side : Bid(Tick Dir. = ‘^’) / Ask(Tick Dir. = ‘v’)
- set bid size =1 / ask side = -1 (focus buying in this paper)
- daily S_t = $\sum$ volume of each trade $*$ side

(2022/03/23-2022/06/23)

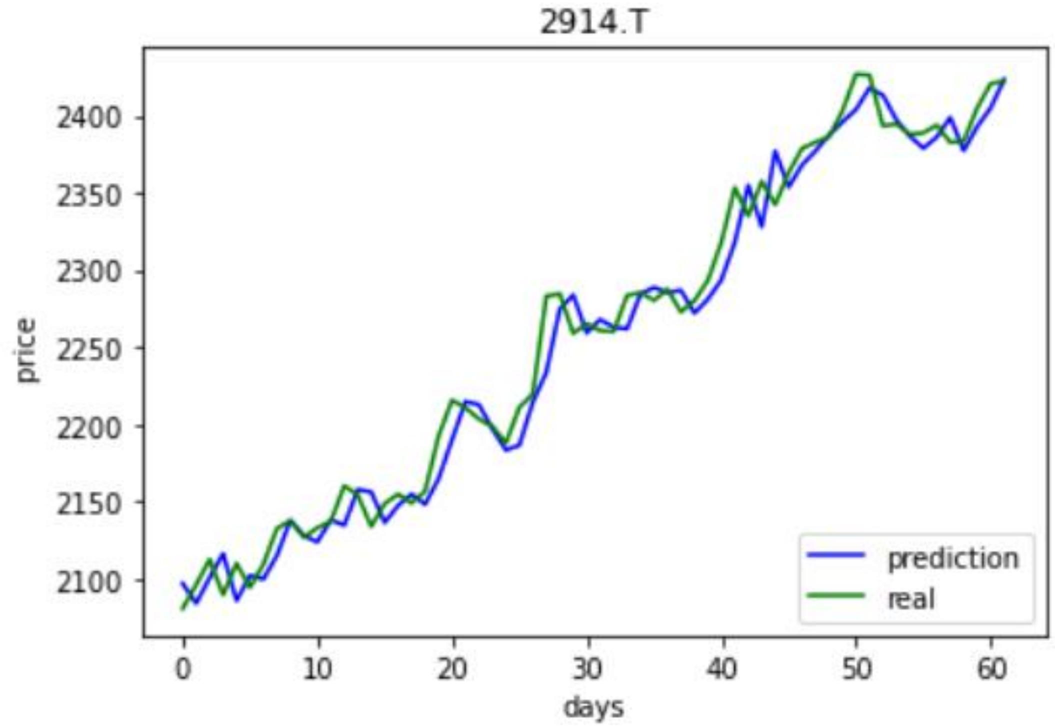
θ, γ

- $P_t = P_{t-1} + \theta S_t + \gamma X_t + \varepsilon_t$, θ>0
- $\mathrm{E}[P_t - P_{t-1}] = \theta S_t + \gamma X_t$,

$where\ P_t$: daily VWAP price

- get θ = 4.3475 $*10^{-6}$, γ = 6.9912

(by scipy package, curve_fit function)



	coef	std err	t	P>|t|	[0.025	0.975]
X1	4.348e-06	1.53e-06	2.844	0.006	1.29e-06	7.41e-06
X2	6.9912	278.227	0.025	0.980	-549.545	563.528
y.L1	-0.1337	0.123	-1.088	0.277	-0.374	0.107

γ, ρ : not significant

Parameter (Summary)

- θ = 4.3475 $*10^{-6}$
- γ = 6.9912
- ρ = -0.13356
- σ_η^2 = 1- ρ^2 = 0.9822
- T = 5
- $\bar{S}$ = 100000

daily trade volume*side

$$P_t - P_{t-1} = \theta S_t + \gamma X_t + \cancel{\varepsilon_t},\ \theta > 0$$

daily VWAP price

daily return rate of ^N225

$$P_t = P_{t-1} + \theta S_t + \gamma X_t + \varepsilon_t, \ \theta>0$$
$$X_t = \rho X_{t-1} + \eta_t, \ \rho \in (-1,1)$$

Result

$$S^*_{T-k} = \overset{①}{\frac{1}{k+1} W_{T-k}} + \overset{②}{\frac{\rho b_{k-1}}{2a_{k-1}} X_{T-k}}$$

	Pt	St	deltaW	del_x	pho*x	Vt	actual cost
1	2414.393445	111292.7473	20000	91292.74734	0.078140896	241263271.8	268704479.6
2	2409.843178	102739.4022	-2823.186835	105562.589	0.095458672	-27353672.71	247585847.5
3	2417.103507	-192791.7643	-38010.71651	-154781.0478	-0.154709378	-275737219.6	-465997649.7
4	2410.441736	151844.0547	39379.8074	112464.2473	0.139874008	189797216	366011246.9
5	2421.377391	-73084.43995	1.609651038	-73086.0496	-0.214978723	-176965010.5	-176965010.5

Note:
The non-negativity restriction was not imposed and was not binding in this realization.

① δW_t: affected by remaining shares. If ρ=0, then it is same as naïve strategy

② δX_t: affected by information. If ρ > 0 and X > 0, then we increase the num of trade size

	Without information	With information (optimal strategy)	Improvement
Expected cost	241456734.6728	241263271.8124	193462.8604
Actual cost	241463185.1412	239338913.8219	2124271.3193

expected of naïve cost :
$$\mathrm{E}_1\left[\sum_{t=1}^{T} P_t \bar{S}/T\right] = P_0\bar{S} + \theta\left(\frac{\bar{S}}{T}\right)^2 \frac{T(T+1)}{2} + \frac{\bar{S}}{T}\frac{\gamma X_1}{1-\rho}\left(T - \frac{\rho-\rho^{T+1}}{1-\rho}\right).$$

$P_t = P_{t-1} + \theta S_t + \gamma X_t + \varepsilon_t$, θ>0
$X_t = \rho X_{t-1} + \eta_t$, ρϵ(-1,1)

Linear price impact with information

- the best-execution strategy with information varies over time as a linear function of remaining shares W_t and information variable X_t

$$S^*_{T-k} = \frac{1}{k+1} W_{T-k} + \frac{\rho b_{k-1}}{2a_{k-1}} X_{T-k}$$

- first term : **naïve strategy**, second term : **adjustment from information**
- ρ = 0, implies X_{T-k}(positive) is unforecastable, no longer affect the strategy
- ρ > 0 (without loss of generality : γ>0), increase the number of shares purchased
  ρ < 0, decrease the number of shares purchased

current execution strategy

- linear market impact **without information**

$$P_t = P_{t-1} + \theta S_t + \varepsilon_t, \quad \theta > 0, \mathrm{E}[\varepsilon_t | S_t, P_{t-1}] = 0$$

$$S^*_{\mathrm{T}-k} = W_{T-k}/(\mathrm{k+1})$$

$$V_{T-k}(P_{T-k-1}, W_{T-k}) = W_{T-k}\left(P_{T-k-1} + \frac{k+2}{2(k+1)}\theta W_{T-k}\right)$$

- linear market impact **with information**

$$P_t = P_{t-1} + \theta S_t + \gamma X_t + \varepsilon_t, \ \theta>0$$

$$X_t = \rho X_{t-1} + \eta_t, \ \rho\epsilon(-1,1)$$

$$S^*_{T-k} = \frac{1}{k+1} W_{T-k} + \frac{\rho b_{k-1}}{2a_{k-1}} X_{T-k}$$

$$V_{T-k}(P_{T-k-1}, X_{T-k}, W_{T-k}) = P_{T-k-1} W_{T-k} + a_k W^2_{T-k} + b_k X_{T-k} W_{T-k} + c_k X^2_{T-k} + d_k$$

$$P_t = P_{t-1} + \theta S_t + \varepsilon_t$$
$$P_t = P_{t-1} + \theta S_t + \gamma X_t + \varepsilon_t$$

model limitations

- there are several important limitations:
  - positive probability of negative prices
  - price impact and information have only permanent effects on prices, which contradicts several recent empirical studies (combination of permanent and temporary effects)

Linear-percentage temporary price impact

- let execution price be comprised of two components: no impact price $\widetilde{P_t}$ and price impact Δ_t

$$P_t = \widetilde{P_t} + \Delta_t$$

- no impact price $\widetilde{P_t}$: plausible and observable proxy for such a price is the midpoint of the bid/ask price
- price dynamics follow geometric Brownian motion (to ensure non-negative price)

$$\widetilde{P_t} = \widetilde{P_{t-1}} * \exp(Z_t)\,,$$
$$where\ Z_t\ is\ a\ normal\ random\ variable$$

- price impact:

$$\Delta_t = (\theta S_t + \gamma X_t)\,\widetilde{P_t}$$
$$X_t = \rho X_{t-1} + \eta_t$$

optimization problem

$$\underset{\{S_t\}}{\text{Min}}\ \mathrm{E}_1\left[\sum_{t=1}^{T} P_t S_t\right] = \underset{\{S_t\}}{\text{Min}}\ \mathrm{E}_1\left[\sum_{t=1}^{T} \tilde{P}_t(1 + \theta S_t + \gamma X_t) S_t\right]$$

$$= \underset{\{S_t\}}{\text{Min}}\ \left\{\mathrm{E}_1\left[\sum_{t=1}^{T} \tilde{P}_t S_t\right] + \mathrm{E}_1\left[\sum_{t=1}^{T} \Delta_t S_t\right]\right\}$$

(1) P_t is guarantee to be non-negative under mild restrictions on Δ_t
(2) the price impact is temporary, moving current price but having no effect on future price
(3) percentage price impact increase linearly with trade size, which is more plausible
(4) implies a natural decomposition of execution costs, decoupling market-microstructure effects from price dynamics

Best execution strategy

$$S^*_{T-k} = \delta_{x,\,k} X_{T-k} + \delta_{w,\,k} W_{T-k} + \delta_{1,\,k}$$

$$V_{T-k}(\tilde{P}_{T-k-1}, X_{T-k}, W_{T-k}) = q\tilde{P}_{T-k-1}[a_k + b_k X_{T-k} + c_k X^2_{T-k}$$
$$+ d_k X_{T-k} W_{T-k} + e_k W_{T-k} + f_k W^2_{T-k}]$$

where:

$$\delta_{x,k} = \frac{q\rho d_{k-1} - \gamma}{2(\theta + qf_{k-1})}, \quad \delta_{w,k} = \frac{qf_{k-1}}{\theta + qf_{k-1}}, \quad \delta_{1,k} = \frac{qe_{k-1} - 1}{2(\theta + qf_{k-1})}$$

$$a_k = \delta_{1,k}(1 + \theta\delta_{1,k}) + q(a_{k-1} + \sigma_\eta^2 c_{k-1}) - q\delta_{1,k}(e_{k-1} - \delta_{1,k} f_{k-1}),$$
$$b_k = q\rho b_{k-1} - \delta_{x,k}(qe_{k-1} - 1),$$
$$c_k = \delta_{x,k}(\theta\delta_{x,k} + \gamma) + q\rho^2 c_{k-1} - q\delta_{x,k}(\rho d_{k-1} - \delta_{x,k} f_{k-1}),$$

$$d_k = \gamma\delta_{w,k} + q\rho d_{k-1}(1 - \delta_{w,k}),$$
$$e_k = \delta_{w,k} + q(1 - \delta_{w,k})e_{k-1},$$
$$f_k = \theta\delta_{w,k}\,.$$
$$q \equiv \mathrm{E}[\exp(Z_t)] = \exp(\mu_z + \sigma^2/2)$$

implementation

- Japan Tobacco Inc., 2914.T
- 2022/06/24 – 2022/06/30(5 days)
- Only discuss ‘buying’
- Step:
  1. Decide parameters
  2. performance of strategy with and without information

$$P_t = \widetilde{P_t} + \Delta_t,$$
$$\widetilde{P_t} = \widetilde{P_{t-1}} * \exp(Z_t)$$
$$\Delta_t = (\theta S_t + \gamma X_t)\,\widetilde{P_t}$$
$$X_t = \rho X_{t-1} + \eta_t$$

Rewrite function

- $P_t = \widetilde{P_t} + \Delta_t$

$$P_t = \widetilde{P_t} + (\theta S_t + \gamma X_t)\, \widetilde{P_t}$$

$$\frac{P_t - \widetilde{P_t}}{\widetilde{P_t}} = \theta S_t + \gamma X_t$$

- S_t : = $\sum$ volume of each trade $*$ side (2022/03/23-2022/06/23)
- X_t : daily return of ^N225 (2022/03/23-2022/06/23)

As mentioned : ρ = -0.13356 (p.10)

$$P_t = \widetilde{P_t} + \Delta_t$$
$$\widetilde{P_t} = \widetilde{P_{t-1}} * \exp(Z_t)$$

$P_t, \widetilde{P_t}$

- P_t : daily VWAP price
- $\widetilde{P_t}$: midpoint of the bid/ask price (daily, from 3 months data)

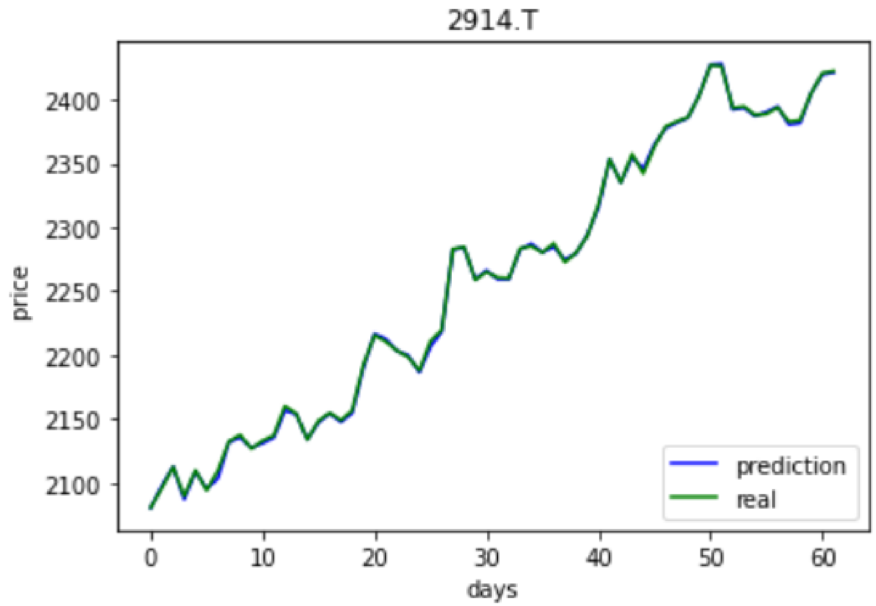
$$\widetilde{P_t} = \frac{\widetilde{P_{t,bid}} + \widetilde{P_{t,ask}}}{2}$$

- $Z_t = \log\left(\frac{\widetilde{P_t}}{\widetilde{P_{t-1}}}\right) \sim N(\mu_z, \sigma_z)$

- get $\mu_z = 2.3649 * 10^{-3}$, ${\sigma^2}_z = 5.5645 * 10^{-5}$

θ, γ

- $\frac{P_t-\widetilde{P_t}}{\widetilde{P_t}} = \theta S_t + \gamma X_t$
- get θ = 1.3235 $*10^{-10}$, γ = 2.7351 $*10^{-2}$
- (by scipy package, curve_fit function)



	coef	std err	t	P>|t|	[0.025	0.975]
X1	1.324e-10	7.03e-11	1.884	0.064	-8.18e-12	2.73e-10
X2	0.0274	0.013	2.139	0.037	0.002	0.053
y.L1	-0.1337	0.123	-1.088	0.277	-0.374	0.107

Parameter

- θ = 1.3235 $*10^{-10}$
- γ = 2.7351 $*10^{-2}$
- ρ = -0.13356
- σ_η^2 = 1- ρ^2 = 0.9822
- T = 5
- $\bar{S}$ = 100000
- μ_z = 2.3649 $*10^{-3}$, $\sigma^2{}_z$= 5.5645 $*10^{-5}$

$$P_t = \widetilde{P_t} + \Delta_t,$$
$$\widetilde{P_t} = \widetilde{P_{t-1}} * \exp(Z_t)$$
$$\Delta_t = (\theta S_t + \gamma X_t)\ \widetilde{P_t}$$
$$X_t = \rho X_{t-1} + \eta_t$$

Result

	Pt	St	deltaW	del_x	del_l	Vt	actual cost
1	2419.989	31514030.88	20095.82404	13394046.95	18099888.11	-7779071091	7626361976
2	2411.96	-25378854.09	-7881717.3	-31066652.29	13569515.49	-81535032486	-6121278405
3	2442.151	-24680674.18	-2016541.054	-31706876.39	9042743.262	-16998248167	-6027393948
4	2432.31	-64448.94082	9333902.161	-13917922.53	4519571.427	45433868057	-156759801.4
5	2408.866	18709946.33	3137007.471	23282129.72	22633861.09	45107340071	4506974822

	Naïve stategy	With information (optimal strategy)	Improvement
Expected cost	289824842.9145	-7779071091.0459	7489246248.1314
Actual cost	242305528.7341	-310115353.7820	67809825.0479

expected of naïve cost : $E\left[P_t \frac{\bar{S}}{T}\right] = \overline{P_t} * \frac{\bar{S}}{T}$

Data of Paper

- $\theta = 5 * 10^{-7}$
- γ = 0, 0.001, 0.0025, 0.005, 0.01
- ρ = -0.5, -0.25, 0.00, 0.25, 0.5
- $\sigma_\eta^2 = 1 - \rho^2$
- T = 20
- $\bar{S}$ = 100000
- $\mu_z = 0$, $\sigma_z = 0.02/\sqrt{13}$

result :

	rho = -0.5	rho = -0.25	rho = 0	rho = 0.25	rho =0.5
gamma=0	13.3058	13.3058	13.3058	13.3058	13.3058
gamma=0.001	12.696	12.6777	12.7366	13.2369	12.9243
gamma=0.0025	9.9019	11.0845	11.2916	11.7987	11.7891
gamma=0.005	0.7618	2.6582	1.1275	4.8861	7.8933
gamma=0.01	-29.8703	-35.194	-26.7913	-21.5339	-20.138

expected cost –no impact cost

$= V_1 - P_0\bar{S}\ (cents/share)$

result of Paper :

γ	$\rho = -0.50$			$\rho = -0.25$			$\rho = 0.00$			$\rho = 0.25$			$\rho = 0.50$		
	S^*	$\bar{S}/T$	Diff.	S^*	$\bar{S}/T$	Diff.	S^*	$\bar{S}/T$	Diff.	S^*	$\bar{S}/T$	Diff.	S^*	$\bar{S}/T$	Diff.
0.0000	13.3058	13.3098	0.0040	13.3058	13.3098	0.0040	13.3058	13.3098	0.0040	13.3058	13.3098	0.0040	13.3058	13.3098	0.0040
	13.3207	13.3260	0.0053	13.2289	13.2329	0.0040	13.2538	13.2575	0.0038	12.9591	12.9584	− 0.0007	13.1136	13.1163	0.0027
	(0.3325)	(0.3349)	(0.0032)	(0.3301)	(0.3325)	(0.0031)	(0.3302)	(0.3325)	(0.0031)	(0.3307)	(0.3331)	(0.0031)	(0.3309)	(0.3333)	(0.0031)
0.0010	12.8778	13.3098	0.4320	12.8933	13.3098	0.4165	12.9195	13.3098	0.3903	12.9590	13.3098	0.3508	13.0228	13.3098	0.2870
	12.9780	13.4086	0.4305	13.0345	13.4556	0.4212	12.9273	13.3059	0.3786	12.9476	13.3056	0.3580	13.3972	13.6819	0.2847
	(0.3308)	(0.3331)	(0.0069)	(0.3306)	(0.3329)	(0.0083)	(0.3319)	(0.3342)	(0.0098)	(0.3292)	(0.3314)	(0.0113)	(0.3308)	(0.3330)	(0.0129)
0.0025	10.6307	13.3098	2.6791	10.7276	13.3098	2.5822	10.8911	13.3098	2.4187	11.1381	13.3098	2.1717	11.5370	13.3098	1.7728
	10.9115	13.5869	2.6754	10.8182	13.4205	2.6023	10.6637	13.1275	2.4638	11.4047	13.5947	2.1900	11.1427	12.8976	1.7550
	(0.3315)	(0.3338)	(0.0166)	(0.3309)	(0.3329)	(0.0197)	(0.3311)	(0.3327)	(0.0236)	(0.3330)	(0.3343)	(0.0275)	(0.3326)	(0.3338)	(0.0319)
0.0050	2.6054	13.3098	10.7044	2.9929	13.3098	10.3169	3.6468	13.3098	9.6630	4.6348	13.3098	8.6750	6.2305	13.3098	7.0793
	2.9075	13.6040	10.6965	2.8609	13.2108	10.3499	3.6529	13.3177	9.6647	4.7800	13.5062	8.7262	6.5647	13.6405	7.0758
	(0.3308)	(0.3317)	(0.0370)	(0.3323)	(0.3331)	(0.0421)	(0.3330)	(0.3322)	(0.0488)	(0.3357)	(0.3333)	(0.0567)	(0.3398)	(0.3363)	(0.0650)
0.0100	− 29.4961	13.3098	42.8059	− 27.9460	13.3098	41.2558	− 25.3304	13.3098	38.6402	− 21.3783	13.3098	34.6881	− 14.9956	13.3098	28.3054
	− 29.4341	13.3893	42.8234	− 28.2093	13.1683	41.3776	− 25.3868	13.2366	38.6233	− 21.3424	13.4958	34.8382	− 15.2465	13.0046	28.2511
	(0.3454)	(0.3343)	(0.1051)	(0.3453)	(0.3344)	(0.1052)	(0.3512)	(0.3368)	(0.1129)	(0.3578)	(0.3393)	(0.1273)	(0.3679)	(0.3432)	(0.1445)

$\rho = 0, \gamma = 0.01$

	Pt	St	deltaW	del_x	del_l	Vt	actual cost
0	50	0	0	0	0	0	0
1	49.97652	9055.584	5000.731	3908.693	146.1606	4973209	452566.576
2	50.17288	7204.875	4787.211	2279.196	138.4676	4517814	361489.3142
3	50.04619	11392.4	4652.805	6608.823	130.7746	4169987	570146.3473
4	50.33836	$\boxed{-2121.38}$	4256.238	-6500.7	123.0817	3602400	-106786.973
5	50.18382	6091.312	4654.82	1321.104	115.3888	3706377	305685.3113
6	50.35467	4273.279	4558.972	-393.389	107.6959	3407218	215179.5645
7	50.0569	7505.358	4579.31	2826.044	100.0031	3201844	375694.9201
8	49.91341	9226.573	4354.138	4780.124	92.3103	2813584	460529.7659
9	49.61123	11998.4	3948.001	7965.786	84.61755	2347334	595255.6394
10	49.9106	11103.94	3216.029	7810.988	76.92485	1741988	554204.3907
11	50.11677	3866.85	2427.133	1370.484	69.23219	1200893	193794.0196
12	50.03306	4229.803	2267.118	1901.145	61.53957	1009757	211629.947
13	51.28595	$\boxed{-17615.8}$	2021.734	-19691.4	53.84698	790302.5	-903443.4348
14	49.83933	18368.02	4827.196	13494.67	46.15444	1695471	915449.6363
15	50.49633	9522.033	2570.231	6913.34	38.46193	765333.4	480827.7351
16	50.96011	11817.15	1179.788	10606.59	30.76947	289879.4	602203.0715
17	51.61484	$\boxed{-5961.72}$	-1479.63	-4505.16	23.07704	-307760	-307712.9959
18	52.32371	$\boxed{-1539.1}$	14.44311	-1568.93	15.38465	851.8286	-80531.54727
19	51.77262	1885.812	791.2216	1086.898	7.692308	82549.9	97633.42654
20	51.91119	-303.381	-14.4489	-5624.39	153.8536	-15816.9	-15748.8662

Problem

- the calibration parameters are not significant, can't effectively represent the overall price dynamics.
- the model fails to cover long-term periods, leading to oscillations in results

Remark 3. Our aim in investigating transaction-triggered price manipulation is to analyze the regularity of market impact models. On this mathematical level, effects as in Figures 4 and 5 are just a theoretical possibility and in fact might appear as curiosities from a practical point of view. However, there is some indication that such oscillatory effects can appear in reality through the interaction of the trading algorithms of several high-frequency traders (HFT). We quote from (CFTC-SEC 2010, page 3):

> ... HFTs began to quickly buy and then resell contracts to each other—generating a "hot-potato" volume effect as the same positions were rapidly passed back and forth. Between 2:45:13 and 2:45:27, HFTs traded over 27,000 contracts, which accounted for about 49 percent of the total trading volume, while buying only about 200 additional contracts net.

Alfonsi, Schied & Slynko (2009)

Part 2

advanced models and implementations

Outline

- review
- the general formulation
- best execution for portfolios
- impose constraints
- Implementation

Review

- Basic Model :

$$Min\ E[\textstyle\sum_{t=1}^{T} P_t S_t]\ , \quad \text{where } \textstyle\sum_{t=1}^{T} S_t = \bar{S}$$

- Price Dynamics :

$$P_t = P_{t-1} + \theta S_t + \varepsilon_t$$

- Dynamic Programming :

$$V_t\left(P_{t-1,} W_t\ \right) = Min\ E_t[P_t S_t + V_{t+1}(P_t, W_{t+1})]$$

- Optimal strategy :

$$\mathrm{S}_1^* = \mathrm{S}_2^* = \cdots = \mathrm{S}_T^* = \bar{S}/\mathrm{T}$$

execution strategy

- linear market impact **without information**

$$P_t = P_{t-1} + \theta S_t + \varepsilon_t$$

$$S_1^* = S_2^* = \cdots = S_T^* = \bar{S}/\mathrm{T}$$

- linear market impact **with information**

$$P_t = P_{t-1} + \theta S_t + \gamma X_t + \varepsilon_t$$

$$S_{T-k}^* = \frac{1}{k+1} W_{T-k} + \frac{\rho b_{k-1}}{2a_{k-1}} X_{T-k}$$

- linear-percentage temporary price impact

$$P_t = \widetilde{P}_t + \Delta_t$$

$$S_{T-k}^* = \delta_{x,\,k} X_{T-k} + \delta_{w,\,k} W_{T-k} + \delta_{1,\,k}$$

The general formulation

- the general approach to minimizing expected execution costs
- objective function

$$Min\; E[\sum_{t=1}^{T} P_t S_t]\;, \quad \text{where } \sum_{t=1}^{T} S_t = \bar{S}$$

- with a more general law of motion

$$P_t = f_t(P_{t-1}, \mathbf{X}_t, S_t, \varepsilon_t),$$

$$\mathbf{X}_t = g_t(\mathbf{X}_{t-1}, \eta_t),$$

$$W_t = W_{t-1} - S_{t-1}, \quad W_1 = \bar{S}, \quad W_{T+1} = 0.$$

- f_t is a general nonlinear and possibly time-varying function
- X_t is a vector of arbitrary dimension which can accommodate multiple factors

Best execution strategy

- k=0

$$V_T(P_{T-1}, \mathbf{X}_T, W_T) = \text{Min } \mathrm{E}_T[P_T S_T] = \mathrm{E}_T[f_T(P_{T-1}, \mathbf{X}_T, W_T, \varepsilon_T) W_T]$$

- k=1

$$V_{T-1}(P_{T-2}, \mathbf{X}_{T-1}, W_{T-1}) = \text{Min } \mathrm{E}_{T-1}[P_{T-1} S_{T-1} + V_T(P_{T-1}, \mathbf{X}_T, W_T)]$$

$$= \text{Min } \mathrm{E}_{T-1}[f_{T-1}(P_{T-2}, \mathbf{X}_{T-1}, S_{T-1}, \varepsilon_{T-1}) S_{T-1} + V_T(f_{T=1}(\cdot), g_T(\cdot), W_{T-1} - S_{T-1})]$$

$$S^*_{T-1} = h_{T-1}(P_{T-2}, \mathbf{X}_{T-1}, W_{T-1})$$

Best execution strategy

- continuing in this fashion

$$V_{T-k}(P_{T-k-1}, \mathbf{X}_{T-k}, W_{T-k}) = \text{Min}\ \mathrm{E}_{T-k}[P_{T-k}S_{T-k} + V_{T-k+1}(P_{T-k}, \mathbf{X}_{T-k+1}, W_{T-k+1})]$$

$$= \text{Min}\ \mathrm{E}_{T-k}[f_{T-k}(P_{T-k-1}, \mathbf{X}_{T-k}, S_{T-k}, \varepsilon_{T-k})S_{T-k} + V_{T-k+1}(f_{T-k}(\cdot), g_{T-k+1}(\cdot), W_{T-k} - S_{T-k})],$$

$$S^*_{T-k} = h_{T-k}(P_{T-k-1}, \mathbf{X}_{T-k}, W_{T-k})$$

- reach k=T-1(starting point)

$$V_1(P_0, \mathbf{X}_1, W_1) = \text{Min}\ \mathrm{E}_1[P_1S_1 + V_2(P_1, \mathbf{X}_2, W_2)] = \text{Min}\ \mathrm{E}_1[f_1(P_0, \mathbf{X}_1, S_1, \varepsilon_1)S_1 + V_2(f_1(\cdot), g_2(\cdot), W_1 - S_1)]$$

$$S^*_1 = h_1(P_0, \mathbf{X}_1, W_1), \quad W_1 = \bar{S}$$

- initial conditions enable us to obtain entire sequence of optimal trades:

$$
\begin{aligned}
S_1^* &= h_1(P_0, \mathbf{X}_1, \bar{S}), \\
S_2^* &= h_2(P_1, \mathbf{X}_2, \bar{S} - S_1^*), \\
&\vdots \\
S_k^* &= h_k\left(P_{k-1}, \mathbf{X}_k, \bar{S} - \sum_{t=1}^{k-1} S_t^*\right), \\
&\vdots \\
S_{T-1}^* &= h_{T-1}\left(P_{T-2}, \mathbf{X}_{T-1}, \bar{S} - \sum_{t=1}^{T-1} S_t^*\right), \\
S_T^* &= \bar{S} - \sum_{t=1}^{T-1} S_t^*.
\end{aligned}
$$

- for certain specifications of the law of motion, computing the optimal control explicitly may be intractable because a closed-form expression for the optimal-value function is not available (propose alternatives in the following slides)

Discretization approach(with grid search)

- discretize possible price as a multiple of some constant d(like 1/8), let **K** be the number of possible values in T periods
- discretize trade size S_t in fixed increments of s shares(like 100 shares), let **J** = $\bar{S}$/s denote the number of round lots that need to be executed initially
- let X_t take on a finite number **N** of possible values

- Under these assumptions, at each time t the optimal-value function Vt(P_{t-1} , X_t , W_t) must be evaluated numerically for KJN possible values. As a result the total memory requirements are of the order O(KJN)

Example

given:

- $\bar{S} = 100000$
- $P_0 = 50$, range : 45~55
- T = 20 periods

get:

- K = 80 (price range / price interval ‘d’)
- J = 1000($\bar{S}$ /execution interval ‘s’)
- N = 10
- KJN = 800000 values of state and control variable in each periods
- total computation : 16 millions
- if V_t takes 10^{-6}s to compute, total computation will be 16 s

not feasible for stocks with high volatility, longer horizons, or a large number of information variables

Approximate dynamic programming

- the optimal-value function is approximated at each stage by a quadratic function
- always yield an analytical but approximate solution

best execution strategy

- let $Y_t = (P_{t-1}, \mathrm{X}_t, W_t)$ denote the state vector at time t
- at k=0 :

compute $V_T(Y_T)$, and we approximate this function with $\widehat{V_T}(Y_T)$ where

$$\hat{V}_T(\mathbf{Y}_T) \equiv \mathbf{Y}_T'\mathbf{Q}_T\mathbf{Y}_T + \mathbf{b}_T'\mathbf{Y}_T$$

and matrix Q_T, vector b_T are selected to minimize :

$$\int_{y_T} (V_T(\mathbf{Y}_T) - \hat{V}_T(\mathbf{Y}_T))^2 \, \mathrm{d}\mathbf{Y}_T$$

- general (T-k) :

find V_{T-k}

$$V_{T-k}(\mathbf{Y}_{T-k}) = \underset{S_{T-k}}{\text{Min}} \ \mathrm{E}_{T-k}[f_{T-k}(P_{T-k-1}, \mathbf{X}_{T-k}, S_{T-k}, \varepsilon_{T-k})S_{T-k}$$
$$+ \hat{V}_{T-k+1}(f_{T-k}(\cdot), g_{T-k+1}(\cdot), W_{T-k} - S_{T-k})].$$

approximate this function

$$\hat{V}_{T-k}(\mathbf{Y}_{T-k}) = \mathbf{Y}'_{T-k}\,\mathbf{Q}_{T-k}\,\mathbf{Y}_{T-k} + \mathbf{b}'_{T-k}\,\mathbf{Y}_{T-k}$$

and matrix Q_{T-k}, vector b_{T-k} are selected to minimize :

$$\int_{y_{T-k}} (V_{T-k}(\mathbf{Y}_{T-k}) - \hat{V}_{T}(\mathbf{Y}_{T-k}))^2 \,\mathrm{d}\mathbf{Y}_{T-k}$$

reason using approximate dynamic programming

- the optimal value functions we have considered are quadratic
- a quadratic approximation can capture a variety of nonlinearities parsimoniously. (more useful than discretization approach)
- the minimization that must be performed at each stage of the dynamic program is considerably more tractable when the optimal-value function is quadratic

Best execution for portfolios

- extend our approach to the multivariate setting in which a portfolio of n stocks must be executed within T periods.
- the important feature : capture cross-stock relations such as cross-autocorrelations.
- price impact may be larger than the sum of the price impact of trading separately.
- if some stocks are negatively correlated, or if the portfolio to be executed includes both purchases and sales, then the execution cost may be lower due to a kind of diversification effect

Basic model – linear price impact case

- $\bar{S} \equiv [\bar{S}_1 \dots \bar{S}_n]'$: the vector of n stocks to be purchased or sold within T periods
- P_t : the vector of prices
- S_t : the vector of shares executed
- W_t : the vector of remaining shares to be executed
- X_t : the vector of m information variables

$$Min\, E\left[\sum_{t=1}^{t=T} P_t{}' S_t\right]$$

subject to

$$\sum_{t=1}^{t=T} S_t = \bar{S}$$

$$W_t = W_{t-1} - S_{t-1}$$

Basic model (portfolio)

$P_t = P_{t-1} + \theta S_t + \gamma X_t + \varepsilon_t$, θ>0

$X_t = \rho X_{t-1} + \eta_t$, ρϵ(-1,1)

$$\mathbf{P}_t = \mathbf{P}_{t-1} + \mathbf{A}\mathbf{S}_t + \mathbf{B}\mathbf{X}_t + \boldsymbol{\varepsilon}_t$$

$$\mathbf{X}_t = \mathbf{C}\mathbf{X}_{t-1} + \boldsymbol{\eta}_t$$

where

A is a positive definite (nxn) matrix

B is an arbitrary (nxm) matrix

ε_t : n vector white noise with mean 0 and covariance matrix $\sum \varepsilon$

η_t : m vector white noise with mean 0 and covariance matrix $\sum \eta$

Best execution strategy

$$\mathbf{S}^*_{T-k} = (\mathbf{I} - \tfrac{1}{2}\mathbf{A}_{k-1}^{-1}\mathbf{A}')\mathbf{W}_{T-k} + \tfrac{1}{2}\mathbf{A}_{k-1}^{-1}\mathbf{B}'_{k-1}\mathbf{C}\mathbf{X}_{T-k}\,,$$

$$\begin{aligned} V_{T-k}(\mathbf{P}_{T-k-1}, \mathbf{X}_{T-k}, \mathbf{W}_{T-k}) = {} & \mathbf{P}'_{T-k-1}\mathbf{W}_{T-k} + \mathbf{W}'_{T-k}\mathbf{A}_k\mathbf{W}_{T-k} \\ & + \mathbf{X}'_{T-k}\mathbf{B}_k\mathbf{W}_{T-k} + \mathbf{X}'_{T-k}\mathbf{C}_k\mathbf{X}_{T-k} + d_k \end{aligned}$$

for $k = 0, 1, \ldots, T-1$, where

$$\mathbf{A}_k = \mathbf{A} - \tfrac{1}{4}\mathbf{A}\mathbf{A}_{k-1}^{-1}\mathbf{A}', \quad \mathbf{A}_0 = \mathbf{A},$$

$$\mathbf{B}_k = \tfrac{1}{2}\mathbf{C}'\mathbf{B}_{k-1}(\mathbf{A}'_{k-1})^{-1}\mathbf{A}' + \mathbf{B}', \quad \mathbf{B}_0 = \mathbf{B}',$$

$$\mathbf{C}_k = \mathbf{C}'\mathbf{C}_{k-1}\mathbf{C} - \tfrac{1}{4}\mathbf{C}'\mathbf{B}_{k-1}(\mathbf{A}'_{k-1})^{-1}\mathbf{B}_{k-1}\mathbf{C}, \quad \mathbf{C}_0 = \mathbf{0},$$

$$d_k = d_{k-1} + \mathrm{E}[\eta'_{T-k}\mathbf{C}_{k-1}\eta_{T-k}], \quad d_0 = 0.$$

Discussion

- it is linear in the two state variables $W_{T-k}\ and\ X_{T-k}$
- unless the matrix A is diagonal, the best-execution strategy for one stock will depend on the parameters and state variables of all the other stocks.
- **if selling in the portfolio, the objective function should be revised :**

$$Min\ E\left[\sum_{t=1}^{t=T}(U_t - P_t)'\, S_t\right]$$

multivariate LPT case

$$P_t = \tilde{P}_t + \Delta_t$$

$$\tilde{P}_t = \tilde{P}_{t-1} \exp(Z_t)$$

$$\Delta_t = (\theta S_t + \gamma X_t)\tilde{P}_t,$$

$$X_t = \rho X_{t-1} + \eta_t$$

$$\mathbf{P}_t = \tilde{\mathbf{P}}_t + \boldsymbol{\Delta}_t,$$

$$\tilde{\mathbf{P}}_t = \exp(\mathbf{Z}_t)\tilde{\mathbf{P}}_{t-1}, \quad \mathrm{vec}(\mathbf{Z}_t) \sim \mathrm{N}(\boldsymbol{\mu}_z, \boldsymbol{\Sigma}_z),$$

$$\boldsymbol{\Delta}_t = \mathrm{diag}[\tilde{\mathbf{P}}_t](\mathbf{A}\mathbf{S}_t + \mathbf{B}\mathbf{X}_t),$$

$$\mathbf{X}_t = \mathbf{C}\mathbf{X}_{t-1} + \boldsymbol{\eta}_t, \quad \boldsymbol{\eta}_t \sim \mathrm{WN}(0_z, \boldsymbol{\Sigma}_\eta),$$

optimal strategy can be calculate recursively :

$$\mathbf{S}^*_{T-k} = \mathbf{L}_k \mathbf{W}_{T-k} + \mathbf{G}_k \mathbf{X}_{T-k} + c_k$$

*We omit these formulae for the sake of brevity - they offer no particular insights or intuition and would lengthen this paper by several pages

Imposing constraints

- In most practical applications, there will be constraints on the kind of execution strategies that institutional investors can follow
- For example, selling stock during purchasing shares
- in practice, buy-programs(sell-programs) will almost be accompanied by non-negativity(non-positivity) constraints

- Monte Carlo simulations : 50000 buy programs samples(LPT case)
- observe the average probability that any trade will be a sale

γ	$\rho =\ -0.50$		$\rho =\ -0.25$		$\rho = 0.00$		$\rho = 0.25$		$\rho = 0.50$	
	Prob. (%)	Size (%)	Prob. (%)	Size (%)	Prob. (%)	Size (%)	Prob. (%)	Size (%)	Prob. (%)	Size (%)
0.0000	0.00	—	0.00	—	0.00	—	0.00	—	0.00	—
	(0.00)		(0.00)		(0.00)		(0.00)	—	(0.00)	—
0.0010	0.00	—	0.00	—	0.00	—	0.00	—	0.00	—
	(0.00)		(0.00)		(0.00)		(0.00)		(0.00)	
0.0025	1.71	1.04	1.55	0.94	1.29	0.88	0.95	0.80	0.48	0.71
	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
0.0050	13.81	7.00	13.43	6.59	12.60	5.92	11.45	5.11	9.21	3.91
	(0.03)	(0.02)	(0.03)	(0.02)	(0.03)	(0.02)	(0.03)	(0.02)	(0.03)	(0.02)
0.0100	28.38	34.21	28.09	33.09	27.48	31.06	26.42	28.05	24.53	22.91
	(0.03)	(0.06)	(0.03)	(0.05)	(0.03)	(0.05)	(0.03)	(0.05)	(0.03)	(0.05)

$$S^*_{T-k} = \delta_{w,\,k} W_{T-k} + \delta_{x,\,k} X_{T-k}$$

imposing constraints is difficult

- assume imposing non negative restrictions

$$S^*_{T-1} = \begin{cases} 0 & \text{if} \quad a_1 W_{T-1} + b_1 X_{T-1} < 0, \\ a_1 W_{T-1} + b_1 X_{T-1} & \text{if} \quad 0 < a_1 W_{T-1} + b_1 X_{T-1} < W_{T-1} \\ W_{T-1} & \text{if} \quad a_1 W_{T-1} + b_1 X_{T-1} > W_{T-1}. \end{cases}$$

- V_{T-k} becomes a piecewise-quadratic function, with 3^k pieces
- when T=20, there are 3^{20} intervals at the last stage
- only feasible for very small numbers of periods T

Closed-form solution with non-negativity constraints

- we present a specification of the law of motion under constraints
- $P_t = P_{t-1} + \theta X_t S_t + \varepsilon_t,\;\; \theta>0$
- $\log X_t = \log X_{t-1} + \eta_t$

$$S^*_{T-k} = a_k W_{T-k},$$

$$V_{T-k}(P_{T-k-1}, X_{T-k}, W_{T-k}) = P_{T-k-1} W_{T-k} + \theta b_k X_{T-k} W^2_{T-k}$$

$$(a_k, b_k) = \begin{cases} \left(1 - \dfrac{1}{2\kappa b_{k-1}},\, 1 - \dfrac{1}{4\kappa b_{k-1}}\right) & \text{if } b_{k-1} \geqslant \dfrac{1}{2\kappa} \\ (0, \kappa b_{k-1}) & \text{if } b_{k-1} < \dfrac{1}{2\kappa} \end{cases}$$

$$\kappa \equiv \mathrm{E}[\exp(\eta_t)] = \mathrm{E}[X_t / X_{t-1}]$$

best execution strategy

$$
\begin{aligned}
S_1^* &= a_{T-1}\bar{S}, \\
S_2^* &= a_{T-2}(1 - a_{T-1})\bar{S}, \\
S_3^* &= a_{T-3}(1 - a_{T-2} - a_{T-1})\bar{S}, \\
&\vdots \\
S_k^* &= a_{T-k}(1 - a_{T-k-1} - \cdots - a_{T-1})\bar{S}, \\
&\vdots \\
S_{T-1}^* &= a_1(1 - a_2 - \cdots - a_{T-1})\bar{S}, \\
S_T^* &= \bar{S} - \sum_{t=1}^{T-1} S_t^*.
\end{aligned}
$$

$$\kappa \equiv \mathrm{E}[\exp(\eta_t)] = \mathrm{E}[X_t/X_{t-1}]$$

discussion

$$(a_k, b_k) = \begin{cases} \left(1 - \frac{1}{2\kappa b_{k-1}}, 1 - \frac{1}{4\kappa b_{k-1}}\right) & \text{if } b_{k-1} \geqslant \frac{1}{2\kappa} \\ (0, \kappa b_{k-1}) & \text{if } b_{k-1} < \frac{1}{2\kappa} \end{cases}$$

- if κ lies in interval (0,1/2], then $a_0 = 1, a_1 = a_2 = \cdots = 0$, With such a negative expected growth rate for the price elasticity, it pays to wait until the very end before trading
- if κ lies in interval (1/2,3/4], $a_0 = 1, a_1 = 1 - \frac{1}{2\kappa}, a_2 = a_3 = \cdots = 0$, trade nothing in the first T-2 periods
- if κ =1, which the best-execution strategy reduces to that of the linear price impact model with no information : $S_1^*= S_2^*=\ldots= S_T^* = \bar{S}/T$, which is naïve strategy
- As κ increases, increasing the opportunity cost of delayed trades, the best execution strategy begins its trading sooner and sooner.

implementation

- Japan Tobacco Inc., 2914.T
- 2022/06/24 – 2022/06/30(5 days)
- Only discuss ‘buying’
- Step:
    1. Decide parameters
    2. performance of strategy

$$P_t = P_{t-1} + \theta X_t S_t + \varepsilon_t,\ \theta>0$$

$$\log X_t = \log X_{t-1} + \eta_t$$

$$\kappa \equiv \mathrm{E}[\exp(\eta_t)] = \mathrm{E}[X_t/X_{t-1}]$$

X_t

- X_t : AR(1) in the logarithm of X_t

$$\log X_t = \log X_{t-1} + \eta_t$$

- set X_t daily return of ^N225 (2022/03/23-2022/06/23)

X_t=(^N225['Close'][t]-^N225['Close'][t-1])/(^N225['Close'][t-1])

- **X_t should bigger than 0 : discard negative X_t**

$$\eta_t = \log X_t - \log X_{t-1} \quad , \quad \kappa \equiv \mathrm{E}[\exp(\eta_t)] = \mathrm{E}[X_t/X_{t-1}]$$

- μ_η = **0.3529**, σ_η = **1.6887**, κ = 0.4609

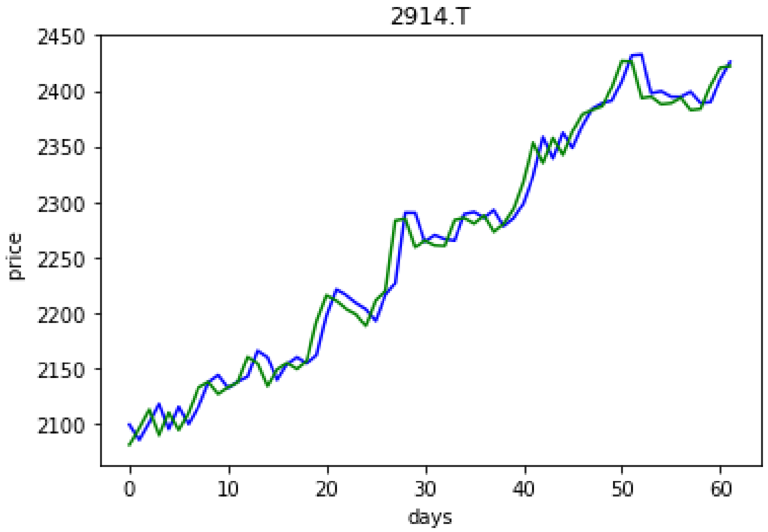
θ

- $P_t - P_{t-1} = \theta S_t X_t + \varepsilon_t,$

$where\ P_t$: daily VWAP price, S_t : daily trade volume

- get θ = $6.6021 * 10^{-5}$, μ_ε = 5.1733, σ_ε = 2.218

	coef	std err	t	P>|t|	[0.025	0.975]
const	5.1733	2.218	2.332	0.023	0.736	9.610
X1	6.602e-05	0.000	0.264	0.793	-0.000	0.001



Parameter

- T = 5, $\bar{S} = 100000$
- θ = 6.6021*10^{-5},
- μ_η = **0.3529**
- σ_η = 1.6887
- μ_ε = **5.1733**
- σ_ε = 2.218
- **κ = 0.4609**

$P_t = P_{t-1} + \theta X_t S_t + \varepsilon_t$, θ>0

$\log X_t = \log X_{t-1} + \eta_t$

$\kappa \equiv \mathrm{E}[\exp(\eta_t)] = \mathrm{E}[X_t / X_{t-1}]$

Result

κ lies in interval (0,1/2],
, it pays to wait until the very end before trading

	Pt	St	Vt	actual cost
1	2429.467576	0	241800017.3	0
2	2437.663979	0	242946802.4	0
3	2443.435504	0	243766972	0
4	2450.781872	0	244345353.1	0
5	2457.519251	100000	245078732.3	245751925.1

	Naïve strategy	Optimal strategy	Improvement(per share)
Expected cost	241800230.1974	241800017.3062	0.0021
Actual cost	244377363.6289	245751925.0784	-13.7456

$$E[\sum P_t \frac{\bar{S}}{T}] = P_0\bar{S} + \theta X_1 \left(\frac{\bar{S}}{T}\right)^2 \frac{\mathrm{T(T+1)}}{2}$$

$P_t = P_{t-1} + \theta X_t S_t + \varepsilon_t$, θ>0
$\log X_t = \log X_{t-1} + \eta_t$
$\kappa \equiv \mathrm{E}[\exp(\eta_t)] = \mathrm{E}[X_t/X_{t-1}]$

Revise the model

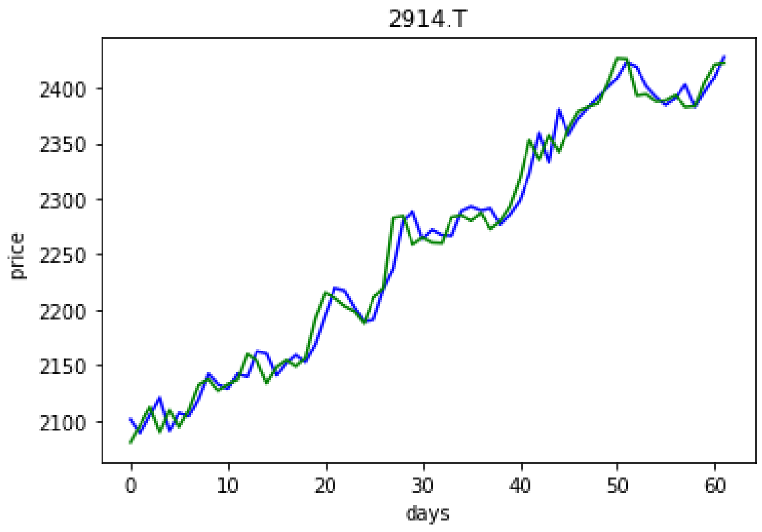
- negative X_t : set X_t be the price of ^N225

  X_t=^N225['Close']

- μ_η = -0.0011, σ_η = 0.0120, κ = 0.9989

- $\theta = 1.4893*10^{-10}$, μ_ε = 4.7612, σ_ε = 2.037

	coef	std err	t	P>|t|	[0.025	0.975]
const	4.7612	2.037	2.337	0.023	0.686	8.836
X1	1.489e-10	5.31e-11	2.803	0.007	4.27e-11	2.55e-10



Parameter

- T = 5, $\bar{S} = 100000$
- θ = 1.4893*10^{-10}
- μ_η = -0.0011
- σ_η = 0.0120
- μ_ε = 4.7612
- σ_ε = 2.037
- κ = 0.9989

$P_t = P_{t-1} + \theta X_t S_t + \varepsilon_t$, θ>0

$\log X_t = \log X_{t-1} + \eta_t$

$\kappa \equiv \mathrm{E}[\exp(\eta_t)] = \mathrm{E}[X_t / X_{t-1}]$

Result

κ is close to 1
, the trading strategy is like naïve strategy

	Pt	St	Vt	actual cost
1	2419.85	19823.8	2.4E+08	47970439.18
2	2424.54	19933.7	1.9E+08	48330195.04
3	2431.05	20021.9	1.5E+08	48674188.35
4	2440.04	20088.2	9.8E+07	49015886.32
5	2446.8	20132.4	4.9E+07	49259911.26

	Naïve strategy	Optimal strategy	Improvement(per share)
Expected cost	241823659.7418	241823624.9916	0.0003
Actual cost	243552004.7933	243556038.9375	-0.0403

$$E[\sum P_t \frac{\bar{S}}{T}] = P_0\bar{S} + \theta X_1 \left(\frac{\bar{S}}{T}\right)^2 \frac{\mathrm{T(T+1)}}{2}$$

Discussion

- solve the oversold / overbought problem
- definition of X_t
- correlation between the price dynamics of ^N225 and 2914.T is close to 0

  (^N225 is not a good information var for 2914.T)
- k still follows the setting of paper

	k=0.4	k=0.6	k=1	k=2
T=1	0	0	20000	70731
T=2	0	0	20000	20732
T=3	0	0	20000	6098
T=4	0	16667	20000	1829
T=5	100000	83333	20000	610

- model isn't flexible，price impact = $\theta X_t S_t$

Limitations – order types

- there is a trade-off between limit and market orders, which generates another dynamic optimization problem
- requires an explicit measure of investors' need for immediacy (**urgency**)
- we can include order type as a control variable and urgency in the objective function, but the problem is computationally intractable.
- two – stage optimization : first optimize the number of shares to be traded within each 30 minute interval, and then perform a second optimization within this 30 minute interval to decide the proportion of market and limit orders to use

Limitations -- risk

four sources of uncertainty:

- the expected cost is itself a function of random initial conditions, and will vary from program to program
- estimation errors of the parameter will be propagated recursively through Bellman's equation
- the law of motion for P and X may suffer from the kind of non-stationarities and time-variation that plague all economic models

Limitation – other objective functions

- while we have focused exclusively on execution costs in this paper, investors are ultimately interested in maximizing the expected utility of their wealth.
- Therefore, the most natural approach to execution costs is to maximize the investor’s expected utility of wealth subject to the law of motion
- although such examples do provide important insights into the economics of transactions costs, they have little to say about minimizing transactions costs in practice.

Limitation – Partial versus general equilibrium

- we assume the parameters and functional form of the law of motion are unaffected by the investor's trades
- However, if a small number of large investors dominate the market, then strategic considerations become more significant, P and X will be directly influenced by these trades

Conclusion $-Min\ E[\sum_{t=1}^{T} P_t S_t]$

- linear market impact without information

$$P_t = P_{t-1} + \theta S_t + \varepsilon_t$$

- linear market impact with information

$$P_t = P_{t-1} + \theta S_t + \gamma X_t + \varepsilon_t$$

- linear percentage temporary(LPT)

$$P_t = \tilde{P}_t + \Delta_t\,, \quad \tilde{P}_t = \tilde{P}_{t-1}\exp(Z_t)\,, \quad \Delta_t = (\theta S_t + \gamma X_t)\tilde{P}_t,$$

- General Formulation (alternative approach : Discretization approach/ approximate dynamic programming)

$$P_t = f_t(P_{t-1}, \mathbf{X}_t, S_t, \varepsilon_t)$$

Conclusion $-Min\ E[\sum_{t=1}^{T} P_t S_t]$

- models for portfolios

$$\mathbf{P}_t = \mathbf{P}_{t-1} + \mathbf{AS}_t + \mathbf{BX}_t + \boldsymbol{\varepsilon}_t$$

$$\mathbf{P}_t = \tilde{\mathbf{P}}_t + \boldsymbol{\Delta}_t, \quad \tilde{\mathbf{P}}_t = \exp(\mathbf{Z}_t)\tilde{\mathbf{P}}_{t-1}, \quad \mathrm{vec}(\mathbf{Z}_t) \sim \mathrm{N}(\boldsymbol{\mu}_z, \boldsymbol{\Sigma}_z), \quad \boldsymbol{\Delta}_t = \mathrm{diag}[\tilde{\mathbf{P}}_t](\mathbf{AS}_t + \mathbf{BX}_t),$$

- imposing constraints
- limitations, extensions, and open questions

Conclusion

- using stochastic dynamic programming, we derived some different strategies that minimize the expected cost of execution
- the best execution strategy is 25% to 40% less than that of the naïve strategy

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